Topology qualifying examination, May 2020

Problem 1. Let $A = [-1, 1] \times 0 \cup 0 \times [-1, 1] \subset \mathbb{R}^2$, and let X be the union of A and the closed disc D in $\mathbb{R}^2$ with the center $(0, 2)$ and the radius equal to 1. Let Y be a 1-dimensional cell complex. Prove that X is not homeomorphic to an open subset of Y.

Problem 2. Let S^2 be the standard unit sphere in $\mathbb{R}^3$, and let $x_1, x_2, \ldots, x_{10} \in S^2$ be pairwise different points. Let J_i be the segment connecting 0 with x_i, where $1 \leqslant i \leqslant 10$. Let $X = S^2 \cup J_1 \cup \ldots \cup J_{10}$. Find the fundamental group of X.

Problem 3. Let D^2 be the unit disc in $\mathbb{R}^2$. Let $p\colon S^1 \longrightarrow S^1$ be an n-sheeted covering space. Let $X_n = D^2/\sim$, where $\sim$ is the following equivalence relation: $x \sim y$ if and only if either $x = y$, or $x, y \in S^1 = \partial D^2$ and $p(x) = p(y)$. Construct explicitly the universal covering space of X_n and show that it is homeomorphic to a subspace of $\mathbb{R}^3$.

Problem 4. Let us consider $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ consisting of points with the last coordinate equal to 0. This leads to a sequence of natural inclusions

$$S^0 \subset S^1 \subset \ldots \subset S^m \subset \ldots .$$

Let S^∞ be the union of this increasing sequence of sets. Suppose that S^∞ is equipped with a topology in such a way that: (i) the induced topology on every S^n, $n \neq \infty$, is the standard one; (ii) every compact subset of S^∞ is contained in S^n for some $n \neq \infty$.

Find the homology groups $H_i(S^\infty)$, where $i = 0, 1, 2, \ldots$.

Problem 5. Let $\Delta^n \subset \mathbb{R}^{n+1}$ be the standard n-simplex, and let

$$\delta_i = \delta_i^n : \Delta^{n-1} \longrightarrow \Delta^n ,$$

where $0 \leqslant i \leqslant n$, be the standard singular $(n-1)$-simplices corresponding to the faces, i.e. $\delta_i(x_0, x_1, \ldots, x_n) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$. Let

$$c_n = \sum_{i=0}^{n} \delta_i .$$

Compute $\partial(c_n)$, simplify your answer, and prove that it cannot be simplified further.